Stability Analysis of The Twisted Superconducting Semilocal Strings

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We study the stability properties of the twisted vortex solutions in the semilocal Abelian Higgs model with a global $\mathbf{SU}(2)$ invariance. This model can be viewed as the Weinberg-Salam theory in the limit where the non-Abelian gauge field decouples, or as a two component Ginzburg-Landau theory. The twisted vortices are characterized by a constant global current $\mathcal{I}$, and for $\mathcal{I} \to 0$ they reduce to the semilocal strings, that is to the Abrikosov-Nielsen-Olesen vortices embedded into the semilocal model. Solutions with $\mathcal{I} \neq 0$ are more complex and, in particular, they are *less energetic* than the semilocal strings, which makes one hope that they could have better stability properties. We consider the generic field fluctuations around the twisted vortex within the linear perturbation theory and apply the Jacobi criterion to test the existence of the negative modes in the spectrum of the fluctuation operator. We find that twisted vortices do not have the homogeneous instability known for the semilocal strings, neither do they have inhomogeneous instabilities whose wavelength is less than a certain critical value. This implies that short enough vortex pieces are perturbatively stable and suggests that small vortex loops could perhaps be stable as well. For longer wavelength perturbations there is exactly one negative mode in the spectrum whose growth entails a segmentation of the uniform vortex into a non-uniform, 'sausage like' structure. This instability is qualitatively similar to the hydrodynamical Plateau-Rayleigh instability of a water jet or to the Gregory-Laflamme instability of black strings in the theory of gravity in higher dimensions.

PACS numbers: 11.10.Lm, 11.27.+d, 12.15.-y, 98.80.Cq

I. INTRODUCTION

Vortices have been the subject of intense studies ever since their discovery by Abrikosov in the context of the Ginzburg-Landau theory of superconductivity and by Nielsen and Olesen in relativistic field theory [1]. They find numerous applications in many domains of physics ranging from high energy physics and cosmology [2] to various branches of condensed matter physics, such as superconductivity [3] and superfluidity [4] models. In these applications the vortices are most often considered within the original model of Abrikosov, Nielsen and Olesen (ANO) [1], which means as solutions of the Abelian Higgs model containing an Abelian vector A_μ coupled to a complex scalar Φ. However, vortices can also exist in others, more general field theory models.

A particular field theory containing an Abelian vector A_μ and two complex scalars Φ_1 and Φ_2 with a global SU(2) invariance, dubbed semilocal model, has been extensively discussed recently [5]. This model can be considered as a 'minimal' generalization of the original ANO theory with the following interesting properties. First, it can be viewed as the special limit of the electroweak theory of Weinberg and Salam in which the weak mixing angle is $\pi/2$ and the non-Abelian gauge field decouples [5]. Secondly, it describes states with a multicomponent order parameter in condensed matter physics, as for example it the two band superconductivity models or in superfluid systems [6]. Finally, it can be regarded as sector of supersymmetric field theories [7] or as part of one of the Grand Unification Theories that could presumably describe cosmic strings in the early Universe [8]. All this implies that the semilocal model is likely to have interesting physical application, which has inspired considerable interest towards this theory and its solutions.

The original ANO vortex can be embedded into the semilocal theory, since identifying its field A_μ with that of the semilocal model and setting also $\Phi_1 = \Phi$ and $\Phi_2 = 0$ solves the field equations of the theory [9]. Such an embedded vortex is called semilocal string and it behaves identically to the original vortex as long as $\Phi_2 = 0$, but its properties can become quite different as soon as the scalar Φ_2 is excited [5]. For example, although the original ANO vortex is stable, fluctuations of Φ_2 render its embedded version unstable in the parameter region where the Higgs boson mass is larger than the vector boson mass [10].

Not all vortices in the semilocal theory should necessarily be of the embedded ANO type. It has been known for some time that more general vortex solutions (sometimes called 'skyrmions') exist in the theory in the special limit where the Higgs boson mass is equal to the vector boson mass [10], [11]. For these solutions Φ_1 and Φ_2 are both non-vanishing. Quite recently vortex solutions

with a similar property have been constructed within the whole parameter region of the theory in which the Higgs boson mass is larger than the vector boson mass [12]. They are characterized by the twist: a z-dependent relative phase $\exp(iKz)$ between Φ_1 and Φ_2. The twist gives rise to a constant global current along the vortex, whose value, $\mathcal{I}$, is a parameter of the solutions. For this reason the solutions are called twisted superconducting semilocal strings, or twisted vortices for short. For $\mathcal{I} \to 0$ they reduce to the semilocal strings but for $\mathcal{I} \neq 0$ they are physically different. In particular, it turns out that solutions with $\mathcal{I} \neq 0$ are *less energetic* as compared to the semilocal strings. This suggests that, since they are energetically favoured, it is predominantly the twisted vortices and not the semilocal strings which should be created in physical processes leading to vortex formation. Therefore, as far as physical applications is concerned, twisted vortices could be more important than the semilocal strings. However, since they exist precisely in the same parameter region where the latter are unstable, the twisted vortices may well be also unstable, which would somewhat delimit their importance. On the other hand, one may argue that they should have better stability properties than the semilocal strings, since they are less energetic.

Motivated by this, we study in this paper the stability of the twisted vortices within the linear perturbation theory by analysing the linearized equations for fluctuations around the twisted vortex background. Decomposing the perturbations into a sum over Fourier modes proportional to $\exp\{i(\omega t+\kappa z+m\varphi)\}$, the variables in the fluctuation equations separate and the equations reduce to a Schrodinger type spectral problem with the eigenvalue ω^2. If this problem admits eigenstates with $\omega^2 < 0$ then the perturbations will be growing in time and so the background will be unstable. In order to find out whether such negative modes exist, we apply the Jacobi criterion [13], which only requires the knowledge of the $\omega = 0$ solutions of the fluctuation equations.

We arrive at the following conclusions. For $\mathcal{I} \neq 0$ the perturbation equations do not admit negative mode solutions independent of z. This means that the twisted vortices do not have the homogeneous instability known for the semilocal strings [10]. Moreover, considering inhomogeneous z-dependent perturbations of the fundamental twisted vortex, we find that the corresponding eigenvalues are always positive, apart from those corresponding to the modes with $m = 0$ and with the wavelength

$$\lambda > \lambda_{\min} = \frac{\pi}{|K|}, \tag{1}$$

where K is the twist parameter of the vortex. It follows then that short vortex pieces obtained by imposing the periodic boundary conditions on the infinite vortex will be perturbatively stable if their length, L, is less then $\lambda_{\min}$. If $L > \lambda_{\min}$ then the vortex will have enough room to ac-

commodate inhomogeneous instability modes whose growth will lead to its fragmentation into a non-uniform, ‘sausage like’ structure characterized by zones of charge accumulation and by an inhomogeneous current density. One needs to go beyond the linearized approximation in order to fully understand the development of this instability. For higher winding number vortices we find additional instabilities corresponding to their splitting into vortices of lower winding number.

Our general conclusion is that the twisted vortices indeed have better stability properties than the semilocal strings since, unlike the latter, they can be stabilized by imposing periodic boundary conditions. This suggests that small stationary vortex loops, if exist, might be stable, which opens intriguing research perspectives that may lead to interesting applications in various domains of physics ranging form condensed matter physics to cosmology.

The rest of the paper is organized as follows. In Sec.II we summarize the essential properties of the twisted vortices. Sec.III contains the analysis of generic perturbations around the twisted vortex background in the linearized approximation: the mode decomposition, separation of variables in the perturbation equations, gauge fixing, and the reduction to a Schrodinger type spectral problem. The existence of the negative modes in the spectrum is demonstrated with the use of the Jacobi criterion in Sec.IV, where these modes are also explicitly constructed and their dispersion relation is determined. The physical manifestations of the instability are discussed in Sec.V, where analogies from other branches of physics are also discussed. We summarize our results in Sec.VI, while the complete system of perturbation equations is presented in the Appendix.

The issue of vortex perturbations is not so often considered in the literature. Although the basic stability pattern for the ANO vortices was understood long ago [14], the systematic stability analysis in this case was carried out only relatively recently [15]. In our analysis we essentially follow the approach of [15], although working in a different gauge.

II. THE TWISTED SEMILOCAL VORTICES

A. The semilocal model

The model considered here is obtained by replacing the complex scalar of the usual Abelian-Higgs model by a doublet of complex scalars, $\mathbf{\Phi}^{\rm tr}=(\Phi_1,\Phi_2)$. This theory, the *semilocal model,* is by construction invariant under the internal symmetry $\mathbf{SU}(2)_{\rm global}\times\mathbf{U}(1)_{\rm local}$. It is worth noting that this semilocal model describes the bosonic sector of the electroweak theory in the limit where

the weak mixing angles is $\pi/2$ and the non-Abelian gauge field decouples [5].

The Lagrangian density in terms of suitably rescaled dimensionless coordinates and fields reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\mathbf{\Phi})^\dagger D^\mu\mathbf{\Phi} - \frac{\beta}{2}\left(\mathbf{\Phi}^\dagger\mathbf{\Phi} - 1\right)^2, \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\mathbf{\Phi} = \partial_\mu\mathbf{\Phi} - \mathrm{i}A_\mu\mathbf{\Phi}$, the spacetime metric signature being $(+,-,-,-)$. The spectrum of this theory consists of a massive vector boson whose mass in the dimensionless units chosen is $m_v = \sqrt{2}$, of two massless Goldstone scalars, and of a Higgs boson with the mass $m_{\mathrm{H}} = \sqrt{\beta}\, m_v$. The fields transform under $\mathbf{SU}(2)_{\mathrm{global}} \times \mathbf{U}(1)_{\mathrm{local}}$ as

$$A_\mu \to A_\mu + \partial_\mu\Lambda(x), \qquad \mathbf{\Phi} \to \mathbf{U}\mathbf{\Phi}, \tag{3}$$

with $\mathbf{U} = \mathrm{e}^{\mathrm{i}\Lambda(x)+\mathrm{i}\theta_a\tau^a}$ where τ_a are the Pauli matrices and θ^a are constant parameters. The Euler-Lagrange equations of motion obtained by varying the Lagrangian (2) read

$$\begin{aligned}\partial^\mu F_{\mu\nu} &= \mathrm{i}\{(D_\nu\Phi)^\dagger\,\Phi - \Phi^\dagger D_\nu\Phi\},\\ D_\mu D^\mu\Phi &= -\beta\left(\Phi^\dagger\Phi - 1\right)\Phi,\end{aligned} \tag{4}$$

where the dagger stands for the hermitian conjugation.

The internal symmetries of the theory give rise to several independently conserved Noether currents, one of which will be important in what follows,

$$J_\mu = \Re(i\Phi_2^* D_\mu\Phi_2). \tag{5}$$

B. The twisted superconducting vortices

Eqs.(4) admit stationary, cylindrically symmetric solutions of vortex type [12]. For these solutions the fields are parametrized in cylindrical coordinates as

$$A_\mu dx^\mu = a_2(\rho)(\Omega\, dt + K\, dz) + N a_1(\rho) d\varphi, \;\; \mathbf{\Phi} = \begin{pmatrix} f_1(\rho)\mathrm{e}^{\mathrm{i}N\varphi} \\ f_2(\rho)\mathrm{e}^{\mathrm{i}(\Omega t + M\varphi + Kz)} \end{pmatrix}, \tag{6}$$

where all functions of ρ are real and N, M are two integer winding numbers. The two real parameters Ω and K can be seen, respectively, as the relative rotation and twist between the two components of the scalar doublet. With this parametrization Eqs.(4) reduce to a set of four non-

linear ordinary differential equations,

$$
\begin{aligned}
&\frac{1}{\rho}\,(\rho a_2')' = 2a_2 f_1^2 + 2(a_2-1)f_2^2,\\
&\rho\left(\frac{a_1'}{\rho}\right)' = 2f_1^2(a_1-1) + 2f_2^2\left(a_1 - \frac{M}{N}\right),\\
&\frac{1}{\rho}(\rho f_1')' = f_1\left(\frac{N^2(a_1-1)^2}{\rho^2} + (K^2-\Omega^2)a_2^2 + \beta\left(f_1^2+f_2^2-1\right)\right),\\
&\frac{1}{\rho}(\rho f_2')' = f_2\left(\frac{(M-Na_1)^2}{\rho^2} + (K^2-\Omega^2)(a_2-1)^2 + \beta\left(f_1^2+f_2^2-1\right)\right),
\end{aligned}
\tag{7}
$$

with $' = \frac{d}{d\rho}$. One can consistently set in these equations $f_2 = a_2 = 0$. The remaining two non-trivial equations for f_1, a_1 reduce then to the ANO system [1] and so the solutions will be the ANO vortices embedded into the semilocal theory. Such embedded solutions are sometimes called semilocal strings [5].

In addition, for $\beta > 1$, Eqs.(7) possess also more general solutions, called twisted vortices, which have $f_2 \neq 0$ and $a_2 \neq 0$ [12]. Their numerical profiles (see Fig.1) show that f_1, a_1 behave qualitatively in the same way as for the ANO vortex, while f_2, a_2 develop non-zero condensate values in the vortex core and tend to zero for $\rho \to \infty$. For a given $\beta > 1$ these solutions comprise a three parameter family labeled by $N = 1, 2, \ldots$, by $M = 0, 1, \ldots N-1$, and by a real parameter $q = \frac{1}{M!} f_2^{(M)}(0)$: the value of the M-th derivative of f_2 at the vortex center. In what follows we shall call q condensate parameter and shall be mainly considering solutions with $M = 0$, so that $q = f_2(0)$. These parameters determine the value of the combination $K^2 - \Omega^2$ which turns out to be positive for all twisted solutions. It is worth noting that the ansatz (6) preserves its structure under Lorentz boosts along the vortex axis – if we assume that (Ω, K) transform as components of a spacetime vector. Since the Lorentz invariant norm of this vector, $K^2 - \Omega^2$, is positive, the vector is spacelike, and so one can boost away its temporal component by passing to the restframe where $\Omega = 0$. On the other hand, the twist K is an essential parameter that cannot be removed, which is why the vortices are called *twisted*. Since the equations are invariant under $K \to -K$, in what follows we can assume without loss of generality that $K > 0$.

Associated to the twist there is a physical parameter: the total current (5) through the vortex cross section,

$$
\mathcal{I} = \int J_z\,\mathrm{d}^2x = \int \mathrm{d}^2x (A_z - K)\Phi_2^*\Phi_2. \tag{8}
$$

This is zero for the embedded ANO solutions and non-zero for the twisted vortices. The current is a function of the condensate parameter q (also of N, M). In the limit $q \to 0$ one has $f_2 \to 0$,

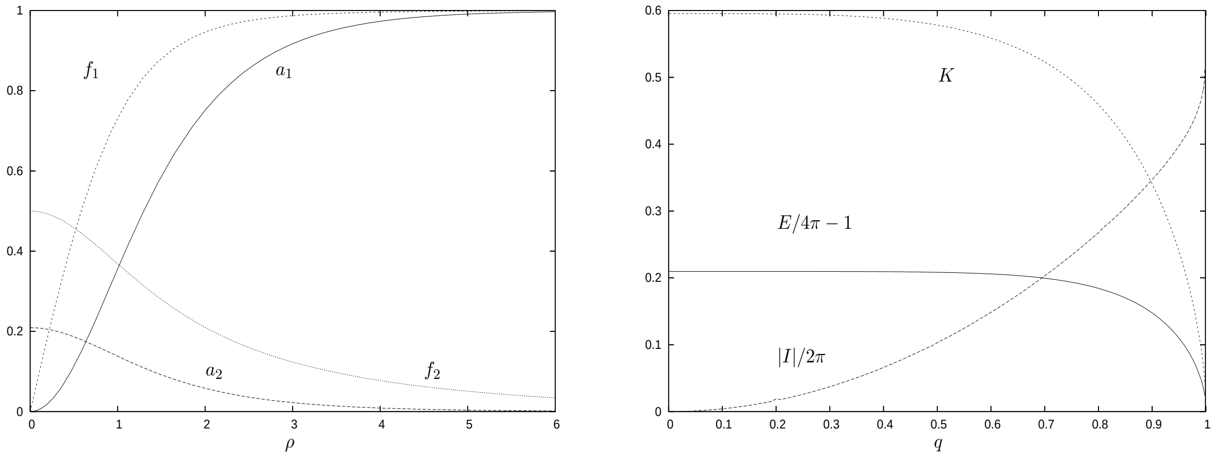

Figure 1: Left: profile functions for the twisted vortex solution with $\beta = 2$, $N = 1$ and $q = 0.5$. Right: the restframe ($\Omega = 0$) vortex energy E, current $\mathcal{I}$, and twist K against the condensate parameter q for the $N = 2$, $M = 0$ twisted solutions with $\beta = 2$.

$a_2 \to 0$, the current vanishes and the twisted vortices reduce to the semilocal strings. For $q \neq 0$ the current is non-zero and it becomes arbitrarily large when $q \to 1$ [12] (although this is not so visible in Fig.1). Interestingly, both the twist K and the vortex energy per unit length, $E = \int T^0_0 d^2x$, *decrease* when the current increases (see Fig.1), for large currents the energy approaching the lower bound $2\pi|N|$ [12]. The twisted current carrying vortices are thus energetically favored when compared to the currentless semilocal strings, and so that one may expect that they could have better stability properties than the latter.

III. PERTURBATIONS OF VORTICES

In order to investigate the stability of twisted vortices, one has to examine the dynamics of their perturbations. Let $\Psi(\vec{r})$ collectively denote the fields of a background vortex solution. Let us consider small perturbations around this background,

$$\Psi(\vec{r}) \to \Psi(\vec{r}) + \delta\Psi(\vec{r})\mathrm{e}^{\mathrm{i}\omega t}.$$

Inserting this into the field equations (4) and linearizing with respect to the perturbations, one can put the linearized equations into a Schrodinger type eigenvalue problem form,

$$(-\Delta + U)\delta\Psi = \omega^2\delta\Psi, \tag{9}$$

where the potential U is determined by the background fields Ψ. If the spectrum of the differential operator in the left hand side of this equation is positive, then all the eigenfrequencies ω are real and so that the background Ψ is linearly stable. If on the other hand there are bound states with $\omega^2 < 0$, the frequency ω will be imaginary and the corresponding perturbation modes will grow in time as $e^{|\omega|t}$ thus indicating the instability of the background.

A. Special fluctuations around the ANO vortex

Let us first briefly consider the $q \to 0$ limit when the background corresponds to the embedded ANO vortex. It is easy to analyze a *particular* type of perturbations around this solution by simply choosing the amplitudes f_2 and a_2 in Eq.(6) to be small, such that the fields read

$$(A_\mu + \delta A_\mu)dx^\mu = \delta a_2(\rho)(\omega\, dt + k\, dz) + N a_1(\rho) d\varphi,$$
$$\mathbf{\Phi} + \delta\mathbf{\Phi} = \begin{pmatrix} f_1(\rho)\mathrm{e}^{\mathrm{i}N\varphi} \\ \delta f_2(\rho)\mathrm{e}^{\mathrm{i}(\omega t + \eta\varphi + kz)} \end{pmatrix}. \tag{10}$$

Here we have replaced Ω, K, M, respectively, par ω, k, η in order to emphasize that these parameters relate to the perturbation and not to the background. Of course, this gives only a special type of perturbations and not the most general one. The perturbation equations are obtained by simply linearizing Eqs.(7) with respect to $f_2 = \delta f_2$ and $a_2 = \delta a_2$. The resulting linear equation for δa_2 decouples and one can show that its only bounded solution is $\delta a_2 = 0$. The equation for δf_2 reads

$$-\frac{1}{\rho}(\rho(\delta f_2)')' + \left(\frac{(Na_1 - \eta)^2}{\rho^2} + \beta\left(f_1^2 - 1\right)\right)\delta f_2 = \varepsilon\delta f_2\,, \tag{11}$$

where f_1 and a_1 refer to the N-th background ANO solution, $\eta = 0, 1, \ldots N-1$, and

$$\varepsilon = \omega^2 - k^2\,. \tag{12}$$

The numerical analysis reveals [10] that if $\beta > 1$ and for $N = 1$, $\eta = 0$ there is a bound state solution of Eq.(11),

$$\delta f_2(\rho) = \Psi_0(\rho), \qquad \varepsilon = \varepsilon_0(\beta) < 0\,, \tag{13}$$

where $\Psi_0(0) \neq 0$, $\Psi_0(\infty) = 0$ and $\int_0^\infty |\Psi_0|^2 \rho d\rho = 1$. The eigenvalue ε_0 is related to the restframe value of the background twist K in the $q \to 0$ limit (see Fig.1) as [12]

$$\varepsilon_0 = -K^2\,. \tag{14}$$

The frequency therefore satisfies the dispersion relation

$$\omega^2 = k^2 - K^2 \tag{15}$$

and so the perturbations

$$\delta\Phi_2 = e^{i\omega t + ikz}\,\Psi_0(\rho) \tag{16}$$

will be growing in time if

$$|k| < |K|. \tag{17}$$

The ANO vortices, when embedded into the semilocal theory, are therefore dynamically unstable [10]. The modes (16) depend on z and so they describe inhomogeneous instabilities, apart from the $k = 0$ mode corresponding to the *homogeneous* instability of the embedded vortex [10]. For

$$|k| > |K| \tag{18}$$

one has $\omega^2 > 0$ and Eq.(16) gives stationary deformations of the ANO vortex. Eq.(10) (with $\delta a_2 = 0$) then describes the ANO background plus the first order correction due to the twist/current. For $k = \pm K$ one has $\omega = 0$ and so Eq.(16) gives zero modes corresponding to static restframe deformations of the ANO vortex by the current.

B. Generic perturbations of twisted vortices

Since the twisted vortices are unstable in the $q \to 0$ limit, when their current vanishes, one can expect that they will presumably be unstable also for $q \neq 0$, at least for $q \ll 1$ when the current is small. However, it is not logically excluded that they may become stable for larger values of the current. Let us therefore consider small fluctuations around the generic twisted vortex configuration,

$$\mathbf{\Phi} = \mathbf{\Phi}^{[0]} + \delta\mathbf{\Phi}^{[0]}, \qquad A_\mu = A_\mu^{[0]} + \delta A_\mu^{[0]}, \tag{19}$$

where $\mathbf{\Phi}^{[0]}$, $A_\mu^{[0]}$ are given by Eq.(6). In order to analyze the dynamics of perturbations it is convenient to make use of the fact that the background configurations are stationary and cylindrically symmetric [12]. Since the corresponding symmetry generators $\partial/\partial t$, $\partial/\partial z$, $\partial/\partial\varphi$ commute between themselves, there exists a gauge where the background fields do not depend on t, z, φ. However, one cannot pass to this gauge using only the local U(1) gauge symmetry that we have

in our disposal, since adjusting *one* gauge function $\Lambda(x)$ in Eq.(3) does not allow to eliminate the phases of the *two* Higgs field components at the same time. However, the following technical trick can be employed.

Let us introduce an auxiliary SU(2) gauge field $W_\mu = \tau^a W^a_\mu$ which is *pure gauge*. Replacing then in Eqs.(2),(4) the covariant derivative as

$$D_\mu \to \mathcal{D}_\mu = D_\mu - \mathrm{i} W^a_\mu \tau^a, \tag{20}$$

the $\mathbf{SU}(2)_{\rm global} \times \mathbf{U}(1)_{\rm local}$ symmetry (3) can be promoted to the $\mathbf{SU}(2)_{\rm local} \times \mathbf{U}(1)_{\rm local}$ gauge transformations:

$$\mathbf{\Phi} \to \mathbf{U\Phi}\ , \quad A_\mu + W_\mu \to \mathbf{U}(A_\mu + W_\mu + i\partial_\mu)\mathbf{U}^{-1}, \tag{21}$$

where $\mathbf{U} = \mathrm{e}^{\mathrm{i}\Lambda(x)+\mathrm{i}\theta_a(x)\tau^a}$. Since W_μ is pure gauge, one can gauge it away and then we return to the previous formulation of the theory. However, allowing for non-zero values of W_μ gives us an additional local SU(2) gauge freedom, which can be useful. Although within the original semilocal model the field W_μ is merely a technical tool, and so we shall have to remove it at the end of calculations, it can be viewed as a physical field if the semilocal model is considered as the limit of the Weinberg-Salam theory.

Let us now apply to Eq.(19) the gauge transformation (21) generated by

$$\mathbf{U} = \begin{pmatrix} e^{-\mathrm{i}N\varphi} & 0 \\ 0 & e^{-\mathrm{i}(\Omega t + Kz + M\varphi)} \end{pmatrix}. \tag{22}$$

This transforms $A^{[0]}_\mu + \delta A^{[0]}_\mu \to A^{[1]}_\mu + \delta A^{[1]}_\mu$ and $\mathbf{\Phi}^{[0]} + \delta\mathbf{\Phi}^{[0]} \to \mathbf{\Phi}^{[1]} + \delta\mathbf{\Phi}^{[1]}$ as well as $W^{[0]}_\mu = 0 \to W^{[1]}_\mu$ where the new background fields

$$\mathbf{\Phi}^{[1]} = \begin{pmatrix} f_1(\rho) \\ f_2(\rho) \end{pmatrix}, \quad W^{[1]}_\mu dx^\mu = \frac{\tau^3}{2}(\Omega dt + K dz + (M-N)d\varphi),$$
$$A^{[1]}_\mu dx^\mu = (a_2(\rho) - \frac{1}{2})(\Omega dt + K dz) + (N(a_1(\rho) - \frac{1}{2}) - \frac{M}{2})d\varphi, \tag{23}$$

while the old and new perturbations are related as

$$\delta A^{[1]}_\mu = \delta A^{[0]}_\mu, \quad \delta\Phi^{[1]}_1 = e^{-\mathrm{i}N\varphi}\delta\Phi^{[0]}_1\,, \quad \delta\Phi^{[1]}_2 = e^{-\mathrm{i}(\Omega t + Kz + M\varphi)}\delta\Phi^{[0]}_2\,. \tag{24}$$

As a result, the background fields depend now only on ρ. The price we pay for this is a new field $W^{[1]}_\mu$ that appears in the new gauge. Moreover, we observe that the azimuthal components of the

vector fields, $A_\varphi^{[1]}$ and $W_\varphi^{[1]}$, do not vanish at $\rho = 0$, which means that the fields are not defined at the symmetry axis. Nevertheless, we can work in this gauge and calculate $\delta A_\mu^{[1]}$, $\delta \mathbf{\Phi}_\mu^{[1]}$, provided that when transformed back to the old gauge where the background fields are globally regular, the perturbations $\delta A_\mu^{[0]}$ and $\delta \mathbf{\Phi}_\mu^{[0]}$ obtained via Eq.(24) will also be regular.

Inserting $\mathbf{\Phi} = \mathbf{\Phi}^{[1]} + \delta\mathbf{\Phi}^{[1]}$ and $A_\mu = A_\mu^{[1]} + \delta A_\mu^{[1]}$ to Eqs.(4), replacing the covariant derivatives D_μ by $\mathcal{D}_\mu$, linearizing with respect to the perturbations and omitting the superscript gives

$$\begin{aligned} &\partial_\mu \partial^\mu \delta A_\nu - \partial_\nu \partial_\mu \delta A^\mu = \\ &= \mathrm{i}\left[(\mathcal{D}_\nu \mathbf{\Phi})^\dagger \delta\mathbf{\Phi} + (\mathcal{D}_\nu \delta\mathbf{\Phi})^\dagger \mathbf{\Phi} - \mathbf{\Phi}^\dagger \mathcal{D}_\nu \delta\mathbf{\Phi} - \delta\mathbf{\Phi}^\dagger \mathcal{D}_\nu \mathbf{\Phi}\right] - 2|\mathbf{\Phi}|^2 \delta A_\nu\,, \\ \mathcal{D}_\mu \mathcal{D}^\mu \delta\mathbf{\Phi} - 2\mathrm{i}\delta A_\mu \mathcal{D}^\mu \mathbf{\Phi} - \mathrm{i}\mathbf{\Phi}\partial_\mu \delta A^\mu &= -\beta\left(2|\mathbf{\Phi}|^2 - 1\right)\delta\mathbf{\Phi} - \beta\delta\mathbf{\Phi}^\dagger \mathbf{\Phi}^2, \end{aligned} \tag{25}$$

where $\mathcal{D}_\mu = \partial_\mu - i(A_\mu + W_\mu)$ and the background fields $\mathbf{\Phi}, A_\mu, W_\mu$ are given by (23).

These equations are invariant under the U(1) gauge transformations

$$\delta A_\mu \to \delta A_\mu + \partial_\mu \delta\Lambda(x), \quad \delta\mathbf{\Phi} \to \delta\mathbf{\Phi} + i\mathbf{\Phi}\delta\Lambda(x), \tag{26}$$

which is the infinitesimal version of the local U(1) gauge symmetry contained in (21).

C. Separation of variables

Since the coefficients in Eqs.(25) depend only on ρ, it is straightforward to separate the variables in these equations by making the Fourier type mode decompositions,

$$\begin{aligned} \delta\Phi_a &= \sum_{\omega,\kappa,m} \cos(\omega t + m\varphi + \kappa z)\left(\phi_a^{\omega,\kappa,m}(\rho) + \mathrm{i}\psi_a^{\omega,\kappa,m}(\rho)\right) \\ &\quad + \sin(\omega t + m\varphi + \kappa z)\left(\pi_a^{\omega,\kappa,m}(\rho) + \mathrm{i}\nu_a^{\omega,\kappa,m}(\rho)\right), \\ \delta A_\mu &= \sum_{\omega,\kappa,m} \xi_\mu^{\omega,\kappa,m}(\rho)\cos(\omega t + m\varphi + \kappa z) + \chi_\mu^{\omega,\kappa,m}(\rho)\sin(\omega t + m\varphi + \kappa z), \end{aligned} \tag{27}$$

where $a = 1, 2$ and $\mu = 1, 2, 3, 4$ with $x^\mu = (t, \rho, z, \varphi)$, respectively. One can similarly decompose the gauge function $\delta\Lambda$ in (26),

$$\delta\Lambda = \sum_{\omega,\kappa,m} (\cos(\omega t + m\varphi + \kappa z)\alpha^{\omega,\kappa,m}(\rho) + \sin(\omega t + m\varphi + \kappa z)\gamma^{\omega,\kappa,m}(\rho)), \tag{28}$$

and so that the gauge transformations (26) assume the form

$$
\begin{aligned}
&\phi_a^{\omega,\kappa,m} \to \phi_a^{\omega,\kappa,m}, && \pi_a^{\omega,\kappa,m} \to \pi_a^{\omega,\kappa,m}, \\
&\nu_a^{\omega,\kappa,m} \to \nu_a^{\omega,\kappa,m} + f_a\gamma^{\omega,\kappa,m}, && \psi_a^{\omega,\kappa,m} \to \psi_a^{\omega,\kappa,m} + f_a\alpha^{\omega,\kappa,m}, \\
&\xi_1^{\omega,\kappa,m} \to \xi_1^{\omega,\kappa,m} + \omega\gamma^{\omega,\kappa,m}, && \chi_1^{\omega,\kappa,m} \to \chi_1^{\omega,\kappa,m} - \omega\alpha^{\omega,\kappa,m} \\
&\chi_2^{\omega,\kappa,m} \to \chi_2^{\omega,\kappa,m} + (\gamma^{\omega,\kappa,m})', && \xi_2^{\omega,\kappa,m} \to \xi_2^{\omega,\kappa,m} + (\alpha^{\omega,\kappa,m})', \\
&\xi_3^{\omega,\kappa,m} \to \xi_3^{\omega,\kappa,m} + \kappa\gamma^{\omega,\kappa,m}, && \chi_3^{\omega,\kappa,m} \to \chi_3^{\omega,\kappa,m} - \kappa\alpha^{\omega,\kappa,m} \\
&\xi_4^{\omega,\kappa,m} \to \xi_4^{\omega,\kappa,m} + m\gamma^{\omega,\kappa,m}, && \chi_4^{\omega,\kappa,m} \to \chi_4^{\omega,\kappa,m} - m\alpha^{\omega,\kappa,m},
\end{aligned}
\tag{29}
$$

where $' = \frac{d}{d\rho}$.

Inserting decompositions (27) to the equations (25), the t, z, φ variables separate and for fixed values of ω, κ, m one obtains a system of 16 ODE's for the 16 real functions $\phi_a^{\omega,\kappa,m}(\rho)$, $\pi_a^{\omega,\kappa,m}(\rho)$, $\nu_a^{\omega,\kappa,m}(\rho)$, $\psi_a^{\omega,\kappa,m}(\rho)$, $\chi_\mu^{\omega,\kappa,m}(\rho)$, $\xi_\mu^{\omega,\kappa,m}(\rho)$. A further inspection reveals that these 16 equations actually split into two independent subsystems of the same size. Specifically, the 8 amplitudes in the left column in (29), whose transformations involve only the gauge functions $\gamma^{\omega,\kappa,m}$, satisfy a closed system of 8 equations. Similarly, the amplitudes in the right column in (29) satisfy a closed system of 8 equations. These two groups of equations become identical upon the replacement

$$
\begin{aligned}
&\pi_a^{\omega,\kappa,m} \to \phi_a^{\omega,\kappa,m}, \qquad \psi_a^{\omega,\kappa,m} \to -\nu_a^{\omega,\kappa,m}, \qquad \xi_2^{\omega,\kappa,m} \to -\chi_2^{\omega,\kappa,m}, \\
&\chi_\mu^{\omega,\kappa,m} \to \xi_\mu^{\omega,\kappa,m}, \quad (\mu = 1, 3, 4).
\end{aligned}
\tag{30}
$$

As a result, without any loss of generality we can restrict our consideration to the 8 equations of the first group. They are explicitly listed in the Appendix, Eqs.(A.1)–(A.8), where we have omitted the superscripts ω, κ, m to simplify the notation. As we shall see later, the two groups of equations describe simply the real and imaginary parts of the perturbations.

D. Gauge fixing and reduction to a Schrodinger form

One can check that Eqs.(A.1)–(A.8) are invariant under the gauge transformations expressed by (29). In addition, they satisfy the differential identity (A.9) expressing the fact that the divergence of the right hand side of the δA_μ equation in Eqs.(25) should vanish, since the divergence of its left hand side vanishes identically. The identity (A.9) expresses one of the 8 equations in terms of the remaining ones, and so only 7 equations are actually independent. As a result, one can exclude one of the equations from consideration.

Let us now specialize to the case of purely magnetic backgrounds by setting

$$\Omega = 0. \tag{31}$$

There is no loss of generality whatsoever in imposing this condition. Indeed, since the twisted vortices with $\Omega \neq 0$ can be obtained from those with $\Omega = 0$ by a Lorentz boost, the same should apply to their perturbations. It is therefore sufficient to consider perturbations in the $\Omega = 0$ case, since Lorentz boosting them will give perturbations for the $\Omega \neq 0$ backgrounds.

Next, we fix the gauge freedom (26),(29) by imposing the temporal gauge condition

$$\delta A_0 = 0 \quad \Leftrightarrow \quad \xi_1 \equiv \xi_1^{\omega,\kappa,m} = 0. \tag{32}$$

This fixes the gauge completely if $\omega \neq 0$, although leaving a residual gauge freedom in the $\omega = 0$ sector expressed by (26) with $\delta\Lambda$ independent on time. Now, equation (A.7) for ξ_1 is the Gauss constraint, and when $\xi_1 = 0$ it assumes the form

$$\omega \left(\chi_2' + \frac{1}{\rho}\,\chi_2 - \frac{m}{\rho^2}\,\xi_4 - \kappa\xi_3 - 2f_1\nu_1 - 2f_2\nu_2 \right) = 0. \tag{33}$$

We use this equation to express ξ_3 in terms of the other variables,

$$\xi_3 = \frac{1}{\kappa} \left(\chi_2' + \frac{\chi_2}{\rho} - \frac{m\xi_4}{\rho^2} - 2(f_1\nu_1 + f_2\nu_2) \right), \tag{34}$$

which is possible as long as $\kappa \neq 0$, and so that from now on we can exclude ξ_3 from the consideration. Since one of the 8 equations is redundant, we can also exclude from the consideration the equation (A.5) for ξ_3. As a result, we now have only 6 equations for the 6 independent field amplitudes ϕ_a, ν_a, χ_2, ξ_4 (see the Appendix for more details).

Remarkably, the elimination of ξ_3 also truncates the residual gauge freedom of time independent gauge transformations. Specifically, the gauge transformation (29) with $\omega = 0$ leave all the equations invariant, in particular the Gauss constraint (33). However, the gauge invariance of the latter is only insured by the overall factor of ω, expressing the fact that Eq.(33) is a total time derivative. When $\omega = 0$, the whole expression in (33) is zero and so that it is gauge invariant. However, the expression between the parenthesis in (33), sometimes called 'strong Gauss constraint' [16], is *not* gauge invariant under (29) for an arbitrary gauge function $\gamma \equiv \gamma^{0,\kappa,m}$ but only if γ satisfies the condition

$$\gamma'' + \frac{1}{\rho}\,\gamma' = \left(\frac{m^2}{\rho^2} + 2(f_1^2 + f_2^2) + \kappa^2 \right) \gamma. \tag{35}$$

Since we have used the strong Gauss constraint to express ξ_3, we have actually reduced the residual gauge freedom to the two parameter family of solutions of this differential equation. It is not then difficult the see that if γ does not vanish identically, then it must be unbounded for small or for large ρ (or in both limits). The behavior of the pure gauge modes produced by this γ is thus incompatible with the boundary conditions at the origin or at infinity (see Eq.(38),(39) below), unless $\gamma = 0$. The only admissible solution of Eq.(35) is therefore $\gamma = 0$, which fixes the residual gauge freedom completely.

We finally arrive at a system of 6 independent second order equations. They are listed in the Appendix, Eqs.(A.14)–(A.19), and they can be rewritten in the form

$$-\Psi'' + \mathbf{U}\Psi = \omega^2\Psi\,. \tag{36}$$

Here Ψ is a 6-component vector whose components are expressed in terms of ϕ_a, ν_a, χ_2, ξ_4 and their first derivatives, and $\mathbf{U}$ is a potential energy matrix determined by the background solution and depending also on κ, m. This determines a linear eigenvalue problem on a half-line, $\rho \in [0,\infty)$, the points $\rho = 0, \infty$ being singular points of the differential equations.

E. Boundary conditions for perturbations

Let us study the local behavior of solutions of Eqs.(36) for small and large ρ. It is convenient to introduce the linear combinations

$$\begin{aligned} \phi_a &= h_a^+ + h_a^-, & \nu_a &= h_a^+ - h_a^-, \\ \xi_4 &= \rho(g^+ - g^-), & \chi_2 &= g^+ + g^-. \end{aligned} \tag{37}$$

Using the asymptotic expansions of the background solutions at small ρ [12] to determine the asymptotic behavior of $\mathbf{U}$ one can construct series solutions of Eqs.(36) in the vicinity of $\rho = 0$. One finds that bounded at $\rho = 0$ solutions behave as

$$h_1^\pm = A_1^\pm \rho^{|N\mp m|} + \ldots, \qquad h_2^\pm = A_2^\pm \rho^{|M\mp m|} + \ldots, \qquad g^\pm = B^\pm \rho^{|m\mp 1|} + \ldots, \tag{38}$$

where $A_a^\pm, B^\pm$ are integration constants and the dots denote the subleading terms.

Let us now consider the asymptotic region, $\rho \to \infty$. Setting the background field amplitudes to their vacuum values, $f_1 = a_1 = 1$, $f_2 = a_2 = 0$, so that the background fields (23) become

pure gauge, Eqs.(A.14)–(A.19) decouple from each other and read (with $\hat{\mathcal{D}}_+$ being defined in the Appendix)

$$\left(-\hat{\mathcal{D}}_+ + \frac{m^2}{\rho^2} + \kappa^2 - \omega^2 + 2\beta\right)\phi_1 = 0,$$
$$\left(-\hat{\mathcal{D}}_+ + \frac{m^2}{\rho^2} + \kappa^2 - \omega^2 + 2\right)\nu_1 = 0,$$
$$\left(-\hat{\mathcal{D}}_+ + \frac{(m \mp 1)^2}{\rho^2} + \kappa^2 - \omega^2 + 2\right)g^\pm = 0,$$
$$\left(-\hat{\mathcal{D}}_+ + \frac{(N - M \mp m)^2}{\rho^2} + (\kappa \pm K)^2 - \omega^2\right)h_2^\pm = 0.$$

The first equation here represents a massive Higgs boson with $m_{\rm H} = \sqrt{2\beta}$, the next three correspond to one longitudinal and two transverse degrees of polarization of a massive vector boson with $m_v = \sqrt{2}$, while the last two equations for $h_2^\pm$ describe a pair of Goldstone particles. The fluctuation equations therefore reproduce correctly the physical spectrum of the semilocal model, and the asymptotic behaviour of the solutions for $\rho \to \infty$ is

$$\phi_1 = \frac{a_1}{\sqrt{\rho}}\,\exp\{-\mu_{\rm H}\rho\} + \ldots, \quad \nu_1 = \frac{a_2}{\sqrt{\rho}}\,\exp\{-\mu_v\rho\} + \ldots,$$
$$g^\pm = \frac{a_\pm}{\sqrt{\rho}}\,\exp\{-\mu_v\rho\} + \ldots, \qquad h_2^\pm = \frac{b^\pm}{\sqrt{\rho}}\,\exp\{-\mu_\pm\rho\} + \ldots\,, \tag{39}$$

where $a_1, a_2, a_\pm, b_\pm$ are integration constants and

$$\mu_{\rm H}^2 = \kappa^2 - \omega^2 + 2\beta, \quad \mu_v^2 = \kappa^2 - \omega^2 + 2, \quad \mu_\pm^2 = (\kappa \pm K)^2 - \omega^2\,. \tag{40}$$

IV. STABILITY TEST

Summarizing the above analysis, we have arrived at the eigenvalue problem (36), and now we wish to find out whether it admits bound state solutions with $\omega^2 < 0$. If exist, such solutions would correspond to unstable modes of the background vortex configuration.

Bound state solutions of Eqs.(36) should be everywhere regular and so they satisfy the boundary conditions (38),(39) at the origin and at infinity. One possibility to proceed would then be to directly integrate Eqs.(36) looking for solutions with $\omega^2 < 0$, which would require solving the boundary value problem for the 6 coupled second order differential equations. Fortunately, there is a faster way, since to know whether the negative modes exist or not it is not actually necessary to construct them explicitly. A simple method that we can apply to reveal their existence is to use the

Jacobi criterion [13] (see [17] for applications of this method for the monopole stability problem). This method essentially relates to the well known fact that the ground state wave function does not oscillate, the first excited state has one node and so on. It follows then that if the zero energy solution of the Schrodinger equation oscillates, then the ground state energy eigenvalue is negative. When applied to multichannel systems as in our case, the Jacobi method gives the following recipe.

Let $\Psi_{(s)}(\rho)$ where $s = 1, \ldots 6$ be 6 linearly independent and regular at the origin solutions of Eqs.(36) with $\omega^2 = 0$, each of them being a 6-component vector $\Psi^I_{(s)}(\rho)$. These solutions can be chosen to satisfy, for example, the conditions $\Psi^I_{(s)}(\rho_0) = \delta^I_s$, where ρ_0 is a point close to the origin. Equally, each of them can be obtained by taking the boundary conditions (38), setting to zero 5 out of the 6 integration constants in (38), and integrating numerically Eqs.(36) with $\omega^2 = 0$ towards large values of ρ. Now, if the determinant

$$\Delta(\rho) = \begin{vmatrix} \Psi^1_{(1)}(\rho) & \ldots & \Psi^1_{(6)}(\rho) \\ \vdots & \ddots & \vdots \\ \Psi^6_{(1)}(\rho) & \ldots & \Psi^6_{(6)}(\rho) \end{vmatrix} \tag{41}$$

vanishes somewhere, then there exists a negative part of the spectrum. Thus we can decide whether the backgrounds is stable or not by simply calculating the determinant (41) – a much easier task than solving the boundary value problem (36) to directly find the eigenvalues. According to [18], the number of instabilities is equal to the number of nodes of $\Delta(\rho)$.

A. Jacobi test in the ANO limit

Before studying the general case, let us consider the limit where $q = f_2 = a_2 = 0$ and the twisted vortices reduce to the embedded ANO solutions. We already know that in this case there is a particular perturbation (10) leading to the eigenvalue problem (11) admitting the negative mode (16). Now we shall be able to take into account all the remaining perturbation modes. The perturbation equations in the ANO limit split into the system of four coupled equations (A.21) plus two decoupled equations (A.22).

The four equations (A.21) do not contain perturbations of the second component of the Higgs field and so they actually describe the dynamics of the perturbed ANO vortex within the original one component ANO model. We therefore expect that for $N = 1$ these equations do not admit bound state solutions with $\omega^2 < 0$, since the ANO vortex is stable within the original ANO theory. However, since for $\beta > 1$ the multivortex configurations of higher winding numbers are unstable

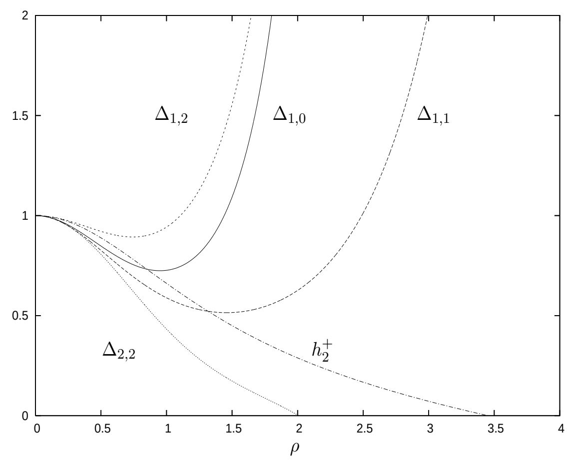

Figure 2: Jacobi determinant $\Delta_{N,m}$ for solutions of Eqs.(A.21) with $\omega = \kappa = 0$ and h_2^+ given by (A.22) with $\omega = m = 0$, $\kappa = K$ in the case of the $\beta = 2$ ANO background with $N = 1$. Since $\Delta_{1,m}$ do not vanish, the $N = 1$ ANO vortex is stable within the original ANO theory. The vanishing of $\Delta_{2,2}$ indicates the splitting instability of the $N = 2$ vortex. The vanishing of h_2^+ shows the instability of the $N = 1$ vortex embedded into the semilocal theory.

with respect to splitting into vortices of lower winding numbers [14],[15], Eqs.(A.21) should admit bound state solutions with $\omega^2 < 0$ for $\beta > 1$ and for $N \geq 2$. We can now test these assertions using the Jacobi criterion. Integrating Eqs.(A.21) with $\omega^2 = 0$ with the regular at the origin boundary conditions (38) to construct their four linearly independent solutions, we calculate the determinant $\Delta(\rho)$ as in Eq.(41). It turns out then that $\Delta(\rho)$ is indeed always positive for $N = 1$, while for $N = 2$ and $\beta > 1$ it changes sign for quadrupole perturbation modes with $m = 2$ (see Fig.2). All this agrees with the expected results.

Let us now consider the two decoupled equations (A.22). They have exactly the same structure as Eq.(11), up to the replacement

$$\delta f_2 \to h_2^{\pm}, \quad \eta \to M \pm m, \quad k \to K \pm \kappa \,. \tag{42}$$

We therefore immediately know that for $N = 1$, in which case $M = 0$, choosing $m = 0$, these equations admit bound state solutions. We can also directly apply the Jacobi criterion to Eqs.(A.22). Setting $\omega = 0$ and also $K + \kappa = 0$ (or $K - \kappa = 0$) and integrating gives a solution for h_2^+ (or for h_2^-) that passes through zero once (see Fig.2) thus showing that there is exactly one bound state.

B. The $N = 1$ twisted vortices

We now have all the necessary ingredients to check the stability of generic twisted vortices. To this end we integrate numerically the system of 6 coupled equations (A.14)–(A.19) with $\omega^2 = 0$ with the regular at the origin boundary conditions (38) to construct the Jacobi determinant. We consider first of all the case of the fundamental $N = 1$ twisted vortex, and for the perturbation parameters we choose the same values which give rise to the instability in the ANO limit: $m = 0$ and $\kappa = K$. The resulting Jacobi determinant $\Delta(\rho)$ for several values of the background condensate parameter q is shown in Fig.3. We find that not only for $q \to 0$ (the ANO limit) but also for all other values in the interval $0 \leq q < 1$ the determinant crosses zero. All these solutions are therefore unstable with respect to axially symmetric perturbations, and we have gone up to $\beta = 10$ to check that the number of instabilities, given by number of zeroes of $\Delta(\rho)$, is exactly one.

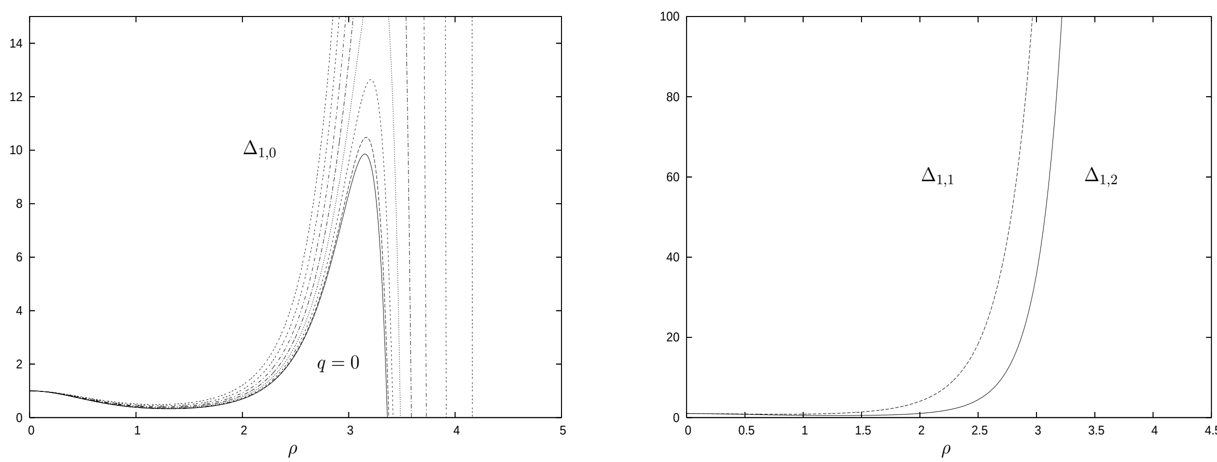

Figure 3: Right: $\Delta_{1,0}$ for $\kappa = K$ for the twisted backgrounds with $\beta = 2$ and $q \in [0; 0.35]$. The vanishing of $\Delta_{1,0}$ indicates the existence of a negative mode. Left: perturbations of the $N = 1$ twisted vortex in the $m > 0$ sector. No instabilities appear. Here $\beta = 2$, $q = 0.5$, $\kappa = K$.

We have also checked the sectors with $m > 0$ but found no instabilities there. In Fig.3 the Jacobi determinants for $m = 1, 2$ are shown, they are everywhere positive. This is not surprising, since increasing m increases the centrifugal energy thus rendering less probable the existence of bound states. Summarizing, all instabilities of the fundamental $N = 1$ twisted vortex reside in the axially symmetric $m = 0$ sector.

C. Solutions with $N > 1$

Vortices with $N > 1$ have additional instabilities. First, they have the same instability in the $m = 0$ sector as the $N = 1$ solutions. This can be checked by calculating the Jacobi determinant for the $m = 0$ perturbations; see Fig.4. In addition, solutions with $N > 1$ can be unstable with respect to splitting to vortices of lower winding number.

The latter instability is present already in the ANO limit (see Fig.2). In the ANO case the existence of this splitting instability can also be inferred from the energy considerations: in the simplest case is suffices to compare the energy of the $N = 2$ vortex to the doubled energy of the $N = 1$ vortex. It turns out then that the former is higher than the latter for $\beta > 1$, which means that the vortex splitting is energetically favoured.

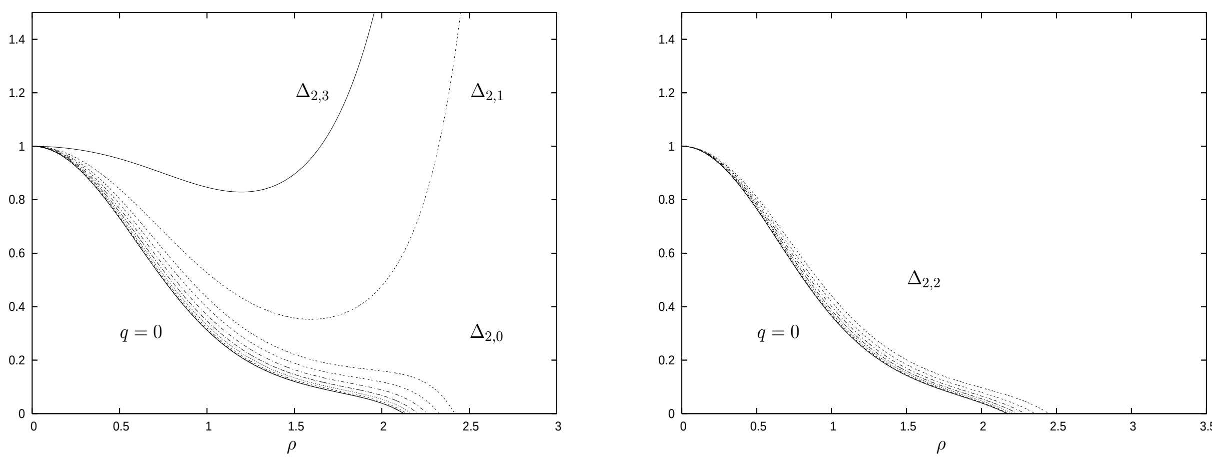

Figure 4: Left: perturbations of the $N = 2$ vortices. Here $\beta = 2$, $\kappa = K$ and $q \in [0; 0.35]$. The same $m = 0$ instability as for the $N = 1$ vortex is detected. $N = 2$ vortices are however stable with respect to perturbations with $m > 0$, $m \neq 2$. Here $\beta = 2$, $\kappa = K$ and $q = 0.5$. Right: they are unstable with respect to the $m = 2$ quadrupole deformations – splitting instability. Here $\beta = 2$, $\kappa = K/2$ and $q \in [0; 0.35]$.

A similar energy argument can also be applied to the twisted vortices. However, the consideration in this case is complicated by the fact that, apart from the winding number N, the twisted vortices also carry additional parameters, in particular the current $\mathcal{I}$. One can also compare the energy of a $N = 2$ twisted vortex, $E(N = 2, \mathcal{I})$, with a sum of energies of two $N = 1$ vortices, $E(N = 1, \mathcal{I}_1) + E(N = 1, \mathcal{I}_2)$, but then one should consider different possibilities for values of $\mathcal{I}_1$ and $\mathcal{I}_2$. Assuming that $\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}$ and choosing $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}/2$ one finds that the splitting is indeed energetically favoured if $\mathcal{I}$ is small, but not for large enough $\mathcal{I}$ [12]. However, one can

also consider the case where $\mathcal{I}_1$ is large and positive, while $\mathcal{I}_2$ is large and negative, their sum $\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}$ being fixed. Since the energy of twisted vortices decreases with the current (see Fig.1), it follows that whatever the value of $\mathcal{I}$ is, one can always adjust $\mathcal{I}_1$ and $\mathcal{I}_2$ such that the splitting will be energetically favorable.

We therefore expect all twisted vortices with $N > 1$ to be unstable with respect to the splitting. This expectation is confirmed by calculating the Jacobi determinant in the $m = 2$ sector, since for all $N = 2$ backgrounds we tested the determinant vanishes somewhere; see Fig.4. We have also checked that $N = 3$ vortices exhibit the splitting instability for $m = 2, 3$.

Summarizing what has been said, $N > 1$ current carrying vortices exhibit the same axially symmetric $m = 0$ instability as the fundamental $N = 1$ vortex. In addition they are unstable with respect to splitting into vortices of lower winding number. We have not found negative modes in sectors with $m = 1$ or for $m > N$.

D. Explicit computation of the eigenvalue

Having revealed the existence of the negative modes in the spectrum of the fluctuation operator, we have also managed to construct them explicitly. Such a construction is considerably more involved than applying the Jacobi criterion, since it requires solving the boundary value problem for the 6 coupled equations (A.14)–(A.19) with the boundary conditions (38) and (39). In addition, one should solve at the same time the 4 equations (7) to generate the background profiles.

It turns out that for the $N = 1$ background Eqs.(A.14)–(A.19) admit two different bound state solutions in the $m = 0$ sector with eigenvalues $\omega^2_{(1)}$ and $\omega^2_{(2)}$, where $\omega^2_{(1)}(\kappa) < 0$ for $\kappa \in (-2K, 0)$ and $\omega^2_{(2)}(\kappa) < 0$ for $\kappa \in (0, 2K)$, also $\omega^2_{(1)}(\kappa) = \omega^2_{(2)}(-\kappa)$. As a result, for each value of $\kappa \in (-2K, 0) \cup (0, 2K)$ there is only one negative mode. For these two bound states all the 6 field amplitudes in Eqs.(A.14)–(A.19) are coupled, but in the limit $q \to 0$ only 2 non-trivial amplitudes remain and the solutions reduce to that for the two decoupled equations (A.22).

Fig.5 shows the dispersion relations $\omega^2_{(1)}(\kappa)$ and $\omega^2_{(2)}(\kappa)$ for several values of q. For $q = 0$ they are given by the formula

$$\omega^2 = (\kappa \pm K)^2 - K^2 \tag{43}$$

which will be explained shortly. For $q > 0$ this formula is no longer exact, but it approximates reasonably well the values of $\omega^2(\kappa)$. (This formula was used to trace the $q = 0.45$ and $q = 0.55$ curves in Fig.5, since the direct calculation of $\omega^2(\kappa)$, performed for $q = 0.15$, becomes rather

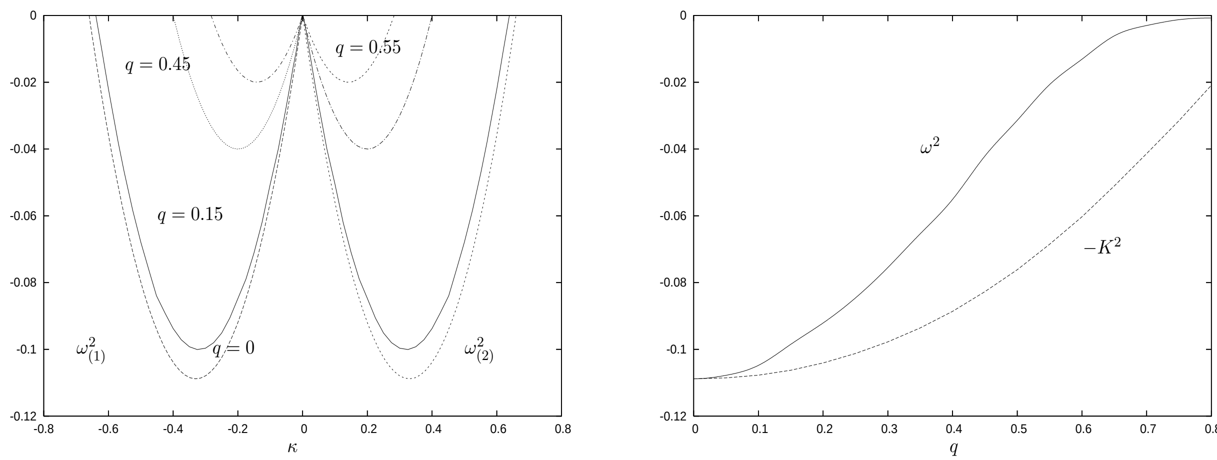

Figure 5: Left: dispersion relations $\omega^2(\kappa)$ for the two bound state solutions of Eqs.(A.14)–(A.19) for the $N = 1$ twisted backgrounds with $\beta = 2$. Right: The minimal value of $\omega^2(\kappa)$ corresponding to $|\kappa| = K$. We see that the instability region bounded from below by the $\omega^2(\kappa)$ curve shrinks when q increases.

time consuming for larger values of q.) The following features of this formula remain true also for $q > 0$. The negative mode exists only for $\kappa \in (-2K, 0) \cup (0, 2K)$. The minimal value of ω^2 is achieved for $\kappa = \pm K$ and this value tends to zero as $q \to 0$, being bounded from below by $-K^2$ (see Fig.5). We observe that the size of the instability region rapidly shrinks for $q \to 1$. The twisted vortices become therefore 'more stable' for large currents. This can be understood by noting that their energy decreases with growing $\mathcal{I}$ and for $\mathcal{I} \to \infty$ approaches the lower bound $2\pi|N|$ [12] and so that less room for instability is left for large $\mathcal{I}$.

Table I: The value of ω^2 for $\kappa = K$, $N = 1$.

q	0	0.05	0.1	0.15	0.2	0.25	0.3
K	0.3298	0.3294	0.3282	0.3259	0.3225	0.3181	0.3126
ω^2	-0.108	-0.107	-0.105	-0.099	-0.092	-0.085	0.076
q	0.35	0.4	0.45	0.5	0.55	0.6	0.65
K	0.3059	0.2976	0.2876	0.2759	0.2617	0.255	0.2259
ω^2	-0.065	-0.055	-0.042	-0.031	-0.0207	-0.013	-0.007

V. PHYSICAL MANIFESTATIONS OF INSTABILITY

Summarizing the above analysis, despite their lower energy, the twisted vortices have essentially the same instabilities as the embedded ANO vortices. However, there is one important difference leading to important consequences: they do not have the uniform spreading instability mode independent of z analogues to the one obtained by setting $k = 0$ in Eq.(16). To understand how this comes about and why this is so important, let us explicitly reconstruct the fields corresponding to the unstable modes of the fundamental $N = 1$ background vortex.

A. Reconstructing the perturbations

Let $\Psi_{(s)}$ with $s = 1, 2$ denote the two linearly independent *normalized* bound state solutions of the eigenvalue problem (36) for a given κ and with $m = 0$, the corresponding eigenvalues being $\omega^2_{(s)}$. Each of them determines the 6 field amplitudes $\phi_a^{(s)}$, $\nu_a^{(s)}$, $\xi_4^{(s)}$, $\chi_2^{(s)}$ in Eqs.(A.14)–(A.19) and these, using (34), determine also $\xi_3^{(s)}$. Let us introduce vectors

$$X^{\pm}_{(s)} = \begin{pmatrix} \phi_a^{(s)} \\ \pm\nu_a^{(s)} \\ \xi_\mu^{(s)} \\ \pm\chi_2^{(s)} \end{pmatrix}, \qquad Y_{\omega,\kappa} = \begin{pmatrix} \phi_a^{\omega,\kappa} \\ \nu_a^{\omega,\kappa} \\ \xi_\mu^{\omega,\kappa} \\ \chi_2^{\omega,\kappa} \end{pmatrix}, \qquad Z_{\omega,\kappa} = \begin{pmatrix} \pi_a^{\omega,\kappa} \\ \psi_a^{\omega,\kappa} \\ \chi_\mu^{\omega,\kappa} \\ \xi_2^{\omega,\kappa} \end{pmatrix}, \tag{44}$$

where $\mu = 3, 4$ and where the components of the vectors $Y_{\omega,\kappa}$ and $Z_{\omega,\kappa}$ are the $m = 0$ amplitudes entering the mode decomposition (27). If $Y_{\omega,\kappa}$ describes a bound state solution of the perturbation equations, then it should be a linear combination of $X^+_{(s)}$, since the latter can be regarded as the basis vectors in the space of bound state solutions. Similarly, if $Z_{\omega,\kappa}$ describes a bound state solution of the second group of perturbation equations, then it should be a linear combination of the corresponding basis vectors. The latter, in view of the symmetry (30), can be chosen to be $X^-_{(s)}$. As a result,

$$Y_{\omega,\kappa} = \sum_s a^\kappa_{(s)} X^+_{(s)}, \qquad Z_{\omega,\kappa} = \sum_s b^\kappa_{(s)} X^-_{(s)}, \tag{45}$$

where $a^\kappa_{(s)}$ and $b^\kappa_{(s)}$ are independent real coefficients. Since the value of ω^2 is fixed for the bound states, the frequency assumes only two values, $\omega = \pm\sqrt{\omega^2}$, and one has similarly

$$Y_{-\omega,\kappa} = \sum_s c^\kappa_{(s)} X^+_{(s)}, \qquad Z_{-\omega,\kappa} = \sum_s d^\kappa_{(s)} X^-_{(s)}. \tag{46}$$

Inserting this to (27) and applying (24) gives the perturbations in the regular gauge,

$$\begin{aligned}
\delta\Phi_1^{[0]} &= e^{iN\varphi}\sum_\kappa\sum_s\left(h_1^{(s)+}\mathcal{Q}^\kappa_{(s)} + h_1^{(s)-}(\mathcal{Q}^\kappa_{(s)})^*\right),\\
\delta\Phi_2^{[0]} &= e^{iM\varphi+iKz}\sum_\kappa\sum_s\left(h_2^{(s)+}\mathcal{Q}^\kappa_{(s)} + h_2^{(s)-}(\mathcal{Q}^\kappa_{(s)})^*\right),\\
\delta A_2^{[0]} &= \sum_\kappa\sum_s i\left((\mathcal{Q}^\kappa_{(s)})^* - \mathcal{Q}^\kappa_{(s)}\right)\chi_2^{(s)},\\
\delta A_\mu^{[0]} &= \sum_\kappa\sum_s\left(\mathcal{Q}^\kappa_{(s)} + (\mathcal{Q}^\kappa_{(s)})^*\right)\xi_\mu^{(s)}, \qquad \mu = 3,4.
\end{aligned} \tag{47}$$

Here $h_a^{(s)\pm} = \frac{1}{2}(\phi_a^{(s)} \pm \nu_a^{(s)})$ and

$$\mathcal{Q}^\kappa_{(s)} = (A^\kappa_{(s)}\exp\{i\omega_{(s)}t\} + B^\kappa_{(s)}\exp\{-i\omega_{(s)}t\})\exp\{i\kappa z\} \tag{48}$$

with $A^\kappa_s = a^\kappa_s - ib^\kappa_s$ and $B^\kappa_s = c^\kappa_s + id^\kappa_s$. Regularity of the perturbations (47) at $\rho = 0$ is guaranteed by the boundary conditions (38).

B. Instabilities of the embedded ANO vortex

Let us consider again the ANO limit. We know that the bound states are then described by solutions of the two decoupled equations (A.22), while Eqs.(A.21) do not have bound state solutions for $N = 1$. As a result, one has $h_1^{(s)\pm} = \chi_2^{(s)} = \xi_\mu^{(s)} = 0$ and, for $s = 1$,

$$h_2^{(1)+} = \Psi_0(\rho), \quad h_2^{(1)-} = 0, \qquad \omega^2_{(1)} = k_+^2 - K^2, \tag{49}$$

while for $s = 2$

$$h_2^{(2)+} = 0, \quad h_2^{(2)-} = \Psi_0(\rho), \qquad \omega^2_{(2)} = k_-^2 - K^2\,, \tag{50}$$

where $\Psi_0(\rho)$ is the same as in Eq.(13) and $k_\pm = \kappa \pm K$. This explains, in particular, the formula (43). The only non-zero perturbation in Eqs.(47) is then

$$\delta\Phi_2 = e^{iKz}\Psi_0(\rho)\sum_\kappa\left(\mathcal{Q}_{\kappa(1)} + \mathcal{Q}^*_{\kappa(2)}\right) \tag{51}$$

or explicitly

$$\begin{aligned}
\delta\Phi_2 = \Psi_0(\rho)\sum_\kappa\Big(&\left[A^\kappa_{(1)}\exp\left\{i\sqrt{\omega^2_{(1)}}\,t\right\} + B^\kappa_{(1)}\exp\left\{-i\sqrt{\omega^2_{(1)}}\,t\right\}\right]\exp\{ik_+z\}\\
&+\left[(A^\kappa_{(2)})^*\exp\left\{-i\sqrt{\omega^2_{(2)}}\,t\right\} + (B^\kappa_{(2)})^*\exp\left\{i\sqrt{\omega^2_{(2)}}\,t\right\}\right]\exp\{-ik_-z\}\Big).
\end{aligned} \tag{52}$$

This expression contains exponentially growing in time terms if $k_\pm^2 - K^2 = \kappa(\kappa \pm 2K) < 0$, which condition is satisfied for $\kappa \in (-2K, 0) \cup (0, 2K)$. Since these terms are proportional to $\exp(\pm ik_\pm z)$, the instability can be viewed as a superposition of standing waves of wavelength $\lambda = 2\pi/k_\pm$ whose amplitude grows in time. The minimal wavelength is $\lambda_{\rm min} = 2\pi/K$ and this suggests that the instability can be removed by imposing the periodic boundary conditions along z-axis with the period $L < \lambda_{\rm min}$. However, this will not remove the particular unstable modes with $k_\pm = 0$ independent on z since it can be considered as periodic with any period. Setting in (52) $k_+ = 0$ or $k_- = 0$ shows that these modes are proportional to each other and so in fact there is only one such homogeneous mode,

$$\delta\Phi_2 = A\,\Psi_0(\rho)\exp(Kt), \tag{53}$$

with $A \in \mathbb{C}$. This mode describes a uniform spreading of the vortex in the x, y plane.

Terms with $|\kappa| > 2K$ in (52) describe waves travelling towards positive and negative values of z with positive or negative frequency. These modes correspond to stationary deformations of the embedded ANO vortex by the twist. In particular, setting $|k_\pm| = K$ and so $\kappa = 0, \pm 2K$ gives zero modes corresponding to static deformations of the vortex by the twist/current,

$$\delta\Phi_2 = \Psi_0(\rho)(A_1\, e^{iKz} + A_2\, e^{-iKz}), \tag{54}$$

two independent modes here corresponding to two possible directions of the current.

It is worth noting that the elementary waves in (52) decouple one from another. In particular, denoting $k = k_+$ one can adjust the parameters $A^\kappa_{(s)}$ and $B^\kappa_{(s)}$ such that the result will agree with Eq.(16).

C. Generic case

Let us now consider perturbations of generic twisted vortices given by (47). As in the ANO limit, there are two linearly independent bound state solutions of Eqs.(A.14)–(A.19) with eigenvalues $\omega^2_{(1)}(\kappa)$ and $\omega^2_{(2)}(\kappa)$, where $\omega^2_{(1)}(\kappa) < 0$ for $-2K < \kappa < 0$ and $\omega^2_{(2)}(\kappa) < 0$ if $0 < \kappa < 2K$ (see Fig.5). The perturbations are therefore given by Eqs.(47) where the sums contain growing in time terms for $\kappa \in (-2K, 0) \cup (0, 2K)$. (When $\kappa \to 0, \pm 2K$ the solutions approach zero modes analogues to (54) and describing static on-shell deformations inside the family of twisted vortex solutions. In the generic case such zero modes contain a term linear in z, as can be seen by differ-

entiating the background fields (6) with respect to K, and so the solutions approach these modes only pointwise and non-uniformly in z.)

The novel feature as compared to the ANO case is that now Eqs.(47) do not contain modes independent of z. This is related to the fact that Eqs.(A.14)–(A.19) no longer decouple into independent subsystems and all of the 6 radial field amplitudes $h_a^{(s)+}$, $\chi_2^{(s)}$, $\xi_4^{(s)}$ are now non-vanishing. Terms with different z-dependence in (47) now turn out to be coupled to each other, which is the consequence of the fact that the background fields (6) depend explicitly on z. For example, it is no longer possible to have, as in Eq.(49), a solution with $h_2^{(1)+} \neq 0$ but with $h_2^{(1)-} = 0$, and so the analog of the solution (53) will read

$$\delta\Phi_2 = A\,\Psi_0(\rho)e^{Kt} + B(\mathcal{I}, t, \rho)e^{\pm 2iKz},$$

where $B(\mathcal{I}, t, \rho) \neq 0$ if $\mathcal{I} \neq 0$. Since all the unstable modes depend on z, the homogeneous negative mode is therefore *absent* for twisted strings

Let us consider the effect of the perturbations on the gauge invariant vortex current. With Eq.(5) one obtains

$$\delta J_z = \Re(i\delta\Phi_2^*\,\partial_z\Phi_2 + i\Phi_a^*\,\partial_z\delta\Phi_2 + 2A_z\Phi_2^*\delta\Phi_2 + f_2^2\delta A_z) \tag{55}$$

and using Eqs.(6),(47) gives the growing part of this expression in the form

$$\delta J_z = f_2(\rho)\sum_{\kappa=-2K}^{2K} e^{\omega t}(\mathcal{A}_\kappa(\rho)\cos(\kappa z) + \mathcal{B}_\kappa(\rho)\sin(\kappa z)), \tag{56}$$

where $\omega = \sqrt{|\omega_{(1)}^2|}$ for $\kappa < 0$ and $\omega = \sqrt{|\omega_{(2)}^2|}$ for $\kappa > 0$. A similar expression can be obtained for δJ_0. Every unstable mode with $\kappa \in (-2K, 2K)$ therefore produces ripples with the wavelength $\lambda = 2\pi/|\kappa|$ on the homogeneous distribution of the background current density. As the amplitude of these ripples grows in time, the current deforms more and more thus tending to evolve into a non-uniform ‘sausage like’ structure characterized by zones of charge accumulation.

Since the instability exists only for a finite range of κ, there is a minimal instability wavelength corresponding to the maximal value of κ,

$$\lambda_{\min} = \frac{\pi}{K}\,. \tag{57}$$

As a result, finite vortex pieces obtained by imposing periodic boundary conditions on the infinite vortex will have no room for inhomogeneous instabilities if their length L is less than $\lambda_{\max}$. Next, as we already know, twisted vortices do not have homogeneous negative modes, which means

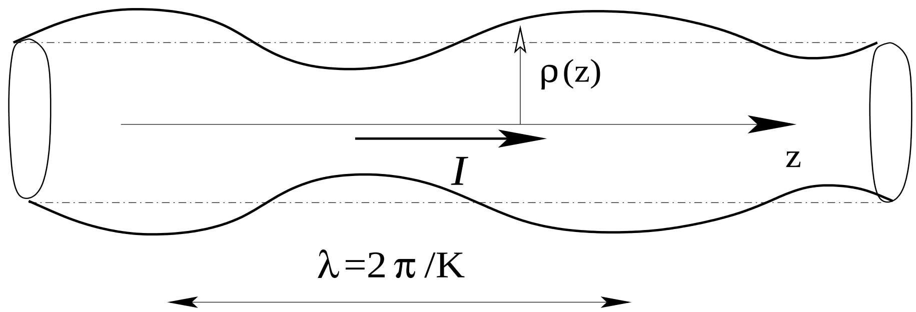

Figure 6: Schematic view of the perturbed vortex current.

modes periodic with any period (for example, the $\kappa = 0$ mode in (56) is homogeneous but not negative). It follows then that short vortex pieces do not have negative modes at all. They are therefore *stable*.

At first view the condition $L < \pi/K$ is impossible, since the background vortex configuration (6) contains $\exp(iKz)$ and so L should be an integer multiple of $2\pi/K$. However, if the semilocal model is viewed as a subsector of the local $\mathbf{SU}(2) \times \mathbf{U}(1)$ theory in the limit where the $\mathbf{SU}(2)$ gauge field decouples, it is possible to use the local gauge transformations (21) not merely as an intermediate technical tool as was done above, but in order to pass to and stay in a physically admissible gauge where the vortex fields do not depend on z. For example, applying to (6) the gauge transformation (21) generated by

$$\mathbf{U} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\mathrm{i}Kz} \end{pmatrix}$$

gives (setting for simplicity $\Omega = M = 0$)

$$A^{[2]}_\mu dx^\mu = K(a_2(\rho) - \frac{1}{2})\, dz + Na_1(\rho)d\varphi, \quad W^{[2]}_\mu dx^\mu = \frac{\tau^3}{2}K dz, \quad \mathbf{\Phi}^{[2]} = \begin{pmatrix} f_1(\rho)\mathrm{e}^{\mathrm{i}N\varphi} \\ f_2(\rho) \end{pmatrix}.$$

This gauge is globally regular and contains no explicit z dependence. One can therefore always work in this gauge and identify z with any period.

To recapitulate, short pieces of twisted $N = 1$ vortices are stable, but they become unstable with respect to fragmentation if their length exceeds a certain critical value. It is worth mentioning an interesting analogy with the well known Plateau-Rayleigh instability of a fluid cylinder in hydrodynamics (see e.g. [19]). The cylinder is stable if only it is short enough, but when it gets longer the surface tension tends to break it up into disjoint pieces. This can easily be seen when water comes out of a kitchen tap: while close to the tap the water jet is homogeneous, far enough

from it the surface ripples appear. This hydrodynamical analogy seems to be not accidental, since the twisted superconducting vortex can effectively be described as a superposition of two charged fluids flowing in the opposite directions [12]. Another interesting analogy that could be mentioned comes from a completely different domain: the Gregory-Laflamme instability of black strings in the gravity theory in higher dimensions [20]. Black strings become unstable with respect to inhomogeneous perturbations if their length exceeds a certain critical value, which can be explained by the tendency of the event horizon to minimize its area thus reducing the black string entropy.

It is worth mentioning that in both of these two examples, and also in our case, the unstable modes exist only in the axially symmetric $m = 0$ sector.

VI. CONCLUDING REMARKS

Summarizing what has been said above, we considered generic field fluctuations around the twisted current carrying vortices in the $\mathbf{SU}(2)_{\rm global}\times\mathbf{U}(1)_{\rm local}$ semilocal field model, or equally in the electroweak theory in the limit where the weak mixing angle is $\pi/2$. By studying the negative modes we have concluded that the twisted vortices exhibit essentially the same instabilities as the embedded ANO vortices, but with one important exception: they do not have the uniform spreading instability. All the negative modes of the fundamental $N = 1$ twisted vortex are non-uniform, with the wavelength bounded from below by $\pi/|K|$, where K is the vortex twist parameter. Since the uniform instability is absent, it follows that short enough vortex pieces should be stable, since they have no room to accommodate the non-uniform longwave instabilities. Longer vortex pieces will be unstable and the instability will lead to their fragmentation into non-uniform objects characterized by zones of charge accumulation and by a non-uniform current density. However, to fully describe this process requires going beyond the linearized approximation.

It is interesting to note that increasing the vortex current $\mathcal{I}$ decreases the twist K (see Fig.1) and so that the minimal instability wavelength $\pi/|K|$ increases, which makes stable longer and longer vortex pieces. In addition, the characteristic time of the instability growth, $2\pi/|\omega|$, also increases, since $|\omega| < |K|$ (see Fig.5). As a result, increasing the current removes more and more of negative modes thus making the vortex ‘more and more stable’.

The non-uniform character of perturbations suggests that if there is a stable configuration to which the perturbed vortex could finally relax, then this configuration should be inhomogeneous. The current should be homogeneous in such a state, though. Specifically, integrating the current

conservation condition $\partial_\mu J^\mu = 0$ over a 3-volume sandwiched between $z = z_1$ and $z = z_2$ planes gives

$$\frac{dQ}{dt} = \mathcal{I}(z_1) - \mathcal{I}(z_2),$$

where Q is the total charge in the volume and $\mathcal{I}(z_1)$ is the total current through the $z = z_1$ plane, similarly for $\mathcal{I}(z_2)$. The condition $\mathcal{I}(z_1) \neq \mathcal{I}(z_2)$ therefore requires that $\frac{dQ}{dt} \neq 0$, and so inhomogeneities of the current can only appear during the dynamical phase when the time derivatives are non-zero. However, if the perturbed vortex finally ends up in an equilibrium state which is stationary, then one will have $\mathcal{I}(z_1) = \mathcal{I}(z_2)$ and so the current will be homogeneous. At the same time, as we already know, the vortex itself cannot be homogeneous in a stable state (since the homogeneous vortex is unstable). We therefore conclude that if a stable stationary state of a current carrying vortex exists, then in this state the current will be constant along the vortex, but the vortex configuration itself will depend non-trivially on z and perhaps also on φ. An interesting problem would be to look for such a stable state via minimizing the energy by keeping the current and also total charge fixed.

Another possibility to have stable objects would be to take short vortex pieces and to close them to make small loops. If exist as stationary field theory solutions, they might be stable. The issue of constructing such solutions is actually quite important, since stationary vortex loops stabilized by centrifugal force (they are called vortons [21]) have been extensively discussed for about last 20 years (see [2] for a review). However, to the best of our knowledge, such objects have never been constructed by resolving the field equations of the underlying field theory, excepting two examples obtained in a *rigid* and not local gauge field theory [22].

It is finally worth noting that the above results are likely to be generalizable in the context of the full electroweak theory for arbitrary values of the weak mixing angle, since the analogs of the twisted vortices are known to exist in this case [23]. It seems plausible that short pieces of these superconducting electroweak vortices could also be stable, and also perhaps small vortex loops.

Acknowledgements

We thank Mark Hindmarsh for interesting discussions. This work was partly supported by the ANR grant NT05-1_42856 'Knots and Vortons'.

Appendix

We derive here the full system of the perturbation equations. The starting point is the mode decomposition (27), inserting which to the fluctuation equations (25) and separating the t, z, φ variables gives, for each set of values of ω, κ, m, a system of 16 ODE's for the 16 radial field amplitudes in (27). These equations split into two independent subsystems of 8 equations each which have exactly the same structure, upon the identification (30). Equations of the first group involve the amplitudes $\phi_a, \nu_a, \xi_1, \chi_2, \chi_3, \chi_4$:

$$
\begin{aligned}
E_1 :=& \left[-\hat{\mathcal{D}}_+ + \frac{N^2(a_1-1)^2+m^2}{\rho^2} + \beta(3f_1^2+f_2^2-1) + (K^2-\Omega^2)a_2^2 + \kappa^2 - \omega^2\right]\phi_1 \\
&+2\beta f_1 f_2\,\phi_2 + 2\left((\Omega\omega - K\kappa)a_2 + \frac{Nm(1-a_1)}{\rho^2}\right)\nu_1 - 2\Omega a_2 f_1 \xi_1 + 2K a_2 f_1\,\xi_3 \\
&+\frac{2Nf_1(a_1-1)}{\rho^2}\,\xi_4 = 0,
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
E_2 :=& \left[-\hat{\mathcal{D}}_+ + \frac{N^2(1-a_1)^2+m^2}{\rho^2} + \beta(f_1^2+f_2^2-1) + (K^2-\Omega^2)a_2^2 + \kappa^2 - \omega^2\right]\nu_1 \\
&+2\left((\Omega\omega - K\kappa)a_2 + \frac{Nm(1-a_1)}{\rho^2}\right)\phi_1 + \omega f_1\,\xi_1 + f_1\chi_2' + \left(2f_1' + \frac{f_1}{\rho}\right)\chi_2 \\
&-\kappa f_1\,\xi_3 - \frac{mf_1}{\rho^2}\,\xi_4 = 0,
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
E_3 :=& \left[-\hat{\mathcal{D}}_+ + \frac{(Na_1-M)^2+m^2}{\rho^2} + \beta(f_1^2+3f_2^2-1) + (K^2-\Omega^2)(a_2-1)^2 + \kappa^2 - \omega^2\right]\phi_2 \\
&+2\beta f_1 f_2\,\phi_1 + 2\left((\Omega\omega - K\kappa)(a_2-1) + \frac{m(M-Na_1)}{\rho^2}\right)\nu_2 + 2\Omega f_2(1-a_2)\,\xi_1 \\
&+2Kf_2(a_2-1)\,\xi_3 + \frac{2f_2(Na_1-M)}{\rho^2}\,\xi_4 = 0,
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
E_4 :=& \left[-\hat{\mathcal{D}}_+ + \frac{(Na_1-M)^2+m^2}{\rho^2} + (K^2-\Omega^2)(a_2-1)^2 + \beta(f_1^2+f_2^2-1) + \kappa^2 - \omega^2\right]\nu_2 \\
&+2\left((\Omega\omega - K\kappa)(a_2-1) + \frac{m(M-Na_1)}{\rho^2}\right)\phi_2 + \omega f_2\,\xi_1 + f_2\chi_2' + \left(2f_2' + \frac{f_2}{\rho}\right)\chi_2 \\
&-\kappa f_2\,\xi_3 - \frac{mf_2}{\rho^2}\,\xi_4 = 0,
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
E_5 :=& \left[-\hat{\mathcal{D}}_+ + \frac{m^2}{\rho^2} + 2(f_1^2+f_2^2) - \omega^2\right]\xi_3 + 4Kf_1a_2\phi_1 + 4Kf_2(a_2-1)\phi_2 \\
&-2\kappa f_1\,\nu_1 - 2\kappa f_2\,\nu_2 + \omega\kappa\,\xi_1 + \kappa\left(\chi_2' + \frac{\chi_2}{\rho}\right) - \frac{\kappa m}{\rho^2}\,\xi_4 = 0,
\end{aligned} \tag{A.5}
$$

$$E_6 := \left[-\hat{\mathcal{D}}_- + 2(f_1^2+f_2^2) + \kappa^2 - \omega^2\right]\xi_4 + 4f_1N(a_1-1)\phi_1 + 4f_2(Na_1-M)\phi_2$$
$$-2mf_1\,\nu_1 - 2mf_2\,\nu_2 + \omega m\,\xi_1 + m\left(\chi_2' - \frac{\chi_2}{\rho}\right) - \kappa m\,\xi_3 = 0, \tag{A.6}$$

$$E_7 := \left[-\hat{\mathcal{D}}_+ + \frac{m^2}{\rho^2} + 2(f_1^2+f_2^2) + \kappa^2\right]\xi_1 + 4\Omega f_1 a_2\,\phi_1 + 4\Omega f_2(a_2-1)\phi_2$$
$$-2\omega f_1\,\nu_1 - 2\omega f_2\,\nu_2 + \omega\left(\chi_2' + \frac{\chi_2}{\rho}\right) - \omega\kappa\,\xi_3 - \frac{\omega m}{\rho^2}\,\xi_4 = 0, \tag{A.7}$$

$$E_8 := \left(\frac{m^2}{\rho^2} + 2(f_1^2+f_2^2) + \kappa^2 - \omega^2\right)\chi_2 + 2(f_1'\nu_1 - f_1\nu_1') + 2(f_2'\nu_2 - f_2\nu_2') + \omega\xi_1'$$
$$-\kappa\xi_3' - \frac{m\xi_4'}{\rho^2} = 0. \tag{A.8}$$

Here $\hat{\mathcal{D}}_\pm := \frac{d^2}{d\rho^2} \pm \frac{1}{\rho}\frac{d}{d\rho}$, the functions f_1, f_2, a_1, a_2 and the constants Ω, K, N, M refer to the background twisted vortex solution.

Not all of these equations are independent. Using the background field equations (7) one can check that the following identity holds,

$$E_8' + \frac{E_8}{\rho} - \frac{m}{\rho^2}\,E_6 - \kappa E_5 + \omega E_7 - 2f_1E_2 - 2f_2E_4 = 0, \tag{A.9}$$

and so one of the equations is redundant.

A good consistency check for the equations is provided by the translational modes. If $A_\mu(x^\alpha), \mathbf{\Phi}(x^\alpha)$ is the background vortex solution written in the gauge (6), then $A_\mu(x^\alpha + X^\alpha), \mathbf{\Phi}(x^\alpha + X^\alpha)$, where X^α is a constant vector, will also be a solution. This implies that the Lie derivatives of $A_\mu(x^\alpha), \mathbf{\Phi}(x^\alpha)$ along X^α,

$$\delta A_\mu = X^\alpha\partial_\alpha A_\mu + A_\alpha\partial_\mu X^\alpha, \quad \delta\mathbf{\Phi} = X^\alpha\partial_\alpha\mathbf{\Phi}, \tag{A.10}$$

should fulfill the linearized field equations. If $X = \frac{\partial}{\partial x}$, the vector generating displacements of the vortex in the x direction, then calculating the Lie derivatives (A.10) and transforming them according to (24) gives the perturbations in the form (27) with $m = 1$, $\omega = \kappa = 0$ and

$$\phi_a = f_a', \quad \nu_1 = -\frac{N}{\rho}\,f_1, \quad \nu_2 = -\frac{M}{\rho}\,f_2,$$
$$\xi_1 = \Omega a_2', \quad \chi_2 = \frac{N}{\rho^2}\,a_1, \quad \xi_3 = Ka_2', \quad \xi_4 = N(a_1' - \frac{1}{\rho}\,a_2). \tag{A.11}$$

Using the background field equations (7) one can check that these fulfill Eqs.(A.1)–(A.8). Choosing similarly $X=\frac{\partial}{\partial z}$ gives $\omega=\kappa=m=0$ and

$$\phi_a=0,\quad \nu_1=0,\quad \nu_2=f_2,\quad \xi_1=\chi_2=\xi_3=\xi_4=0, \tag{A.12}$$

which also solves the equations.

Using again Eqs.(7) one can check that Eqs.(A.1)–(A.8) are invariant under the gauge transformations (29),

$$\phi_a\to\phi_a,\;\; \nu_a\to\nu_a+f_a\gamma,\;\; \xi_1\to\xi_1+\omega\gamma,\;\; \chi_2\to\chi_2+\gamma',\;\; \xi_3\to\xi_3+\kappa\gamma,\;\; \xi_4\to\xi_4+m\gamma\,, \tag{A.13}$$

where $\gamma=\gamma(\rho)$. In order to fix this gauge symmetry we impose the temporal gauge condition $\delta A_0=0$ by setting $\xi_1=0$. We also specialize to the case of purely magnetic backgrounds, $\Omega=0$. Eq.(A.7) can then be resolved with respect to ξ_3 (see Eq.(34)). At the same time one can exclude from consideration Eq.(A.5) containing derivatives of ξ_3, since we know that one of the equations in the system is redundant. Inserting then (34) into the remaining equations (A.1)–(A.4),(A.6),(A.8) gives the 6 independent second order equations for 6 field amplitudes ϕ_a, ν_a, χ_2, ξ_4,

$$\begin{aligned}&\left[-\hat{\mathcal{D}}_+ +\frac{N^2(a_1-1)^2+m^2}{\rho^2}+\beta(3f_1^2+f_2^2-1)+K^2a_2^2+\kappa^2-\omega^2\right]\phi_1+2\beta f_1f_2\phi_2\\&+2\left(\frac{Nm(1-a_1)}{\rho^2}-\kappa Ka_2-\frac{2Ka_2f_1^2}{\kappa}\right)\nu_1-\frac{4Ka_2f_1f_2}{\kappa}\,\nu_2+\frac{2Kf_1a_2}{\kappa}\left(\chi_2'+\frac{\chi_2}{\rho}\right)\\&+\frac{2f_1}{\rho^2}\left(N(a_1-1)-\frac{Kma_2}{\kappa}\right)\xi_4=0\,,\end{aligned} \tag{A.14}$$

$$\begin{aligned}&\left[-\hat{\mathcal{D}}_+ +\frac{N^2(1-a_1)^2+m^2}{\rho^2}+\beta(f_1^2+f_2^2-1)+K^2a_2^2+2f_1^2+\kappa^2-\omega^2\right]\nu_1\\&+2\left(\frac{Nm(1-a_1)}{\rho^2}-K\kappa a_2\right)\phi_1+2f_1f_2\,\nu_2+2f_1'\,\chi_2=0\,,\end{aligned} \tag{A.15}$$

$$\begin{aligned}&\left[-\hat{\mathcal{D}}_+ +\frac{(Na_1-M)^2+m^2}{\rho^2}+\beta(f_1^2+3f_2^2-1)+K^2(a_2-1)^2+\kappa^2-\omega^2\right]\phi_2+2\beta f_1f_2\,\phi_1\\&+\frac{4K(1-a_2)f_1f_2}{\kappa}\,\nu_1+2\left(\frac{m(M-Na_1)}{\rho^2}+K\kappa(1-a_2)+\frac{2K(1-a_2)f_2^2}{\kappa}\right)\nu_2\\&+\frac{2Kf_2(a_2-1)}{\kappa}\left(\chi_2'+\frac{\chi_2}{\rho}\right)+\frac{2f_2}{\rho^2}\left(Na_1-M+\frac{Km(1-a_2)}{\kappa}\right)\xi_4=0\,,\end{aligned} \tag{A.16}$$

$$\left[-\hat{\mathcal{D}}_{+} + (\frac{(M-Na_1)^2+m^2}{\rho^2} + \beta(f_1^2+f_2^2-1) + K^2(1-a_2)^2 + 2f_2^2 + \kappa^2 - \omega^2\right]\nu_2$$
$$+\,2\left(\frac{m(M-Na_1)}{\rho^2} + K\kappa(1-a_2)\right)\phi_2 + 2f_1f_2\,\nu_1 + 2f_2'\,\chi_2 = 0\,, \tag{A.17}$$

$$\left[-\hat{\mathcal{D}}_{-} + \frac{m^2}{\rho^2} + 2f_2^2 + 2f_1^2 + \kappa^2 - \omega^2\right]\xi_4 \quad + 4Nf_1(a_1-1)\phi_1 + 4f_2(Na_1-M)\phi_2 - \frac{2m}{\rho}\,\chi_2 = 0\,, \tag{A.18}$$

$$\left[-\hat{\mathcal{D}}_{+} + \frac{m^2+1}{\rho^2} + 2f_1^2 + 2f_2^2 + \kappa^2 - \omega^2\right]\chi_2 + 4f_1'\,\nu_1 + 4f_2'\,\nu_2 - \frac{2m}{\rho^2}\,\xi_4 = 0. \tag{A.19}$$

As explained in the main text, there is no gauge freedom left in this system. These equations are invariant under

$$m\to -m,\quad \kappa\to-\kappa,\quad \omega^2\to\omega^2,\quad \phi_a\to\phi_a,\quad \nu_a\to-\nu_a,\quad \xi_4\to\xi_4,\quad \chi_2\to-\chi_2\,. \tag{A.20}$$

Taking suitable linear combinations of the 6 field amplitudes ϕ_a, ν_a, χ_2, ξ_4 and their first derivatives one can build a 6-component vector Ψ using which the 6 equations (A.14)–(A.19) can be rewritten in the form of a Schrodinger-type eigenvalue problem (36).

In the ANO limit Eqs.(A.14)–(A.19) split into two independent subsystems. The first subsystem is obtained by setting in Eqs.(A.14),(A.15),(A.18),(A.19) $f_2 = a_2 = 0$:

$$\left[-\hat{\mathcal{D}}_{+} + \frac{N^2(a_1-1)^2+m^2}{\rho^2} + \beta(3f_1^2-1) + \kappa^2 - \omega^2\right]\phi_1 + \frac{2N(1-a_1)}{\rho^2}\,(m\nu_1 - f_1\xi_4) = 0\,,$$
$$\left[-\hat{\mathcal{D}}_{+} + \frac{N^2(1-a_1)^2+m^2}{\rho^2} + \beta(f_1^2-1) + 2f_1^2 + \kappa^2 - \omega^2\right]\nu_1 + \frac{2Nm(1-a_1)}{\rho^2}\,\phi_1 + 2f_1'\,\chi_2 = 0\,,$$
$$\left[-\hat{\mathcal{D}}_{-} + \frac{m^2}{\rho^2} + 2f_1^2 + \kappa^2 - \omega^2\right]\xi_4 + 4Nf_1(a_1-1)\phi_1 - \frac{2m}{\rho}\,\chi_2 = 0\,,$$
$$\left[-\hat{\mathcal{D}}_{+} + \frac{m^2+1}{\rho^2} + 2f_1^2 + \kappa^2 - \omega^2\right]\chi_2 + 4f_1'\,\nu_1 - \frac{2m}{\rho^2}\,\xi_4 = 0. \tag{A.21}$$

Since these equations do not contain perturbations of the second component of the Higgs field, they actually describe the dynamics of the perturbed ANO vortex within the original ANO model.

Eqs.(A.16),(A.17) in the ANO limit reduce to two decoupled equations for the perturbations of the second component of the Higgs field $h_2^{\pm} = \phi_2 \pm \nu_2$

$$\left[-\hat{\mathcal{D}}_{+} + \frac{(Na_1 - M \mp m)^2}{\rho^2} + \beta(f_1^2-1) + (K\pm\kappa)^2 - \omega^2\right]h_2^{\pm} = 0. \tag{A.22}$$

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